Symmetric Spaces Toolkit

SFB/TR12
Langeoog, Nov. 1st–7th 2007

H. Sebert*, S. Mandt†

Contents

1 Lie Groups and Lie Algebras 2
  1.1 Matrix Lie Groups ................................. 2
  1.2 Lie Group Homomorphisms ......................... 3
  1.3 Example: Parametrization of $SO(3, \mathbb{R})$ .......... 3
  1.4 Example: The “covering” $SU(2) \to SO(3)$ .......... 4
  1.5 The Derivative of the Exponential Mapping ......... 4

2 Coset Spaces and Homogenous Spaces 5
  2.1 Existence of Local Coordinates ..................... 6
  2.2 The Tangent Space ................................. 7

3 Symmetric Spaces 7
  3.1 The Cartan Decomposition .......................... 8
  3.2 The Exponential Map for Symmetric Spaces .......... 8
  3.3 The Exponential Map and Geodesics ................. 9

A A Variational Formula for time-dependent Vector Fields 10

B Acknowledgements 10

C Notation 11

*holger.sebert@ruhr-uni-bochum.de
†mandt@thp.uni-koeln.de
1 Lie Groups and Lie Algebras

1.1 Matrix Lie Groups

A subset $G$ of $GL(n, \mathbb{C})$ that is topologically closed and also closed under the group operations (i.e. if $A, B \in G$ then $AB^{-1} \in G$ as well) is called a matrix Lie group. It can be shown (see e.g. [Co]) that such a matrix Lie group is automatically a differentiable sub-manifold of $GL(n, \mathbb{C})$. Note that this definition also includes zero-dimensional matrix groups like $\{\text{Id, } -\text{Id}\} \subset GL(n)$.

In this article all matrix Lie groups are supposed to be reductive, which – in our context – means that for each element $A$ of the group its adjoint $A^\dagger = A^T$ is also an element of the group.\(^1\)

In the context of matrix Lie groups we have the usual exponential map

$$\exp : \mathbb{C}^{n \times n} \to GL(n, \mathbb{C}), \quad X \mapsto e^X = \sum_{k=0}^\infty \frac{X^k}{k!}.$$  \hspace{1cm} (1)

For a given matrix Lie group $G$ we consider the set

$$\mathfrak{g} = \{X \in GL(n) \mid e^{tX} \in G \text{ for all } t \in \mathbb{R}\},$$

and call it the Lie algebra of $G$. In this definition the map $t \mapsto e^{tX}$ is a curve in $G$ which passes through the identity $\text{Id} \in G$. Since

$$\left.\frac{d}{dt}\right|_{t=0} e^{tX} = X$$

we can think of $\mathfrak{g}$ as being the set of tangent vectors at the identity $\text{Id} \in G$, i.e.

$$\mathfrak{g} = T_{\text{Id}}G.$$

Together with the commutator $[X, Y] = XY - YX$ one can show (see for example [Ha]) that $\mathfrak{g}$ is also an Lie algebra in the abstract sense.\(^2\)

Examples of Lie groups and their Lie algebras

- $G = SL(n, \mathbb{C}) = \{A \in \mathbb{C}^{n \times n} \mid \det A = 1\}$, $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C}) = \{X \in \mathbb{C}^{n \times n} \mid \text{tr } X = 0\}$.

\(^1\)This is not the abstract definition of reductivity, which is more complicated (see for example [Hu]). However, it can be shown that a matrix Lie group which fulfills the definition of reductivity given here is also reductive in the abstract sense.

\(^2\)Recall that an (abstract) Lie algebra is a vector space with an additional product structure that is bilinear, skew-symmetric, and fulfills the Jacobi identity, i.e. $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$. 

2
\[ G = SU(n) = \{ AA^\dagger = \text{Id} \} \cap \{ \det A = 1 \}, \quad g = su(n) = \{ X + X^\dagger = 0 \} \cap \{ \text{tr} X = 0 \}. \]

### 1.2 Lie Group Homomorphisms

A **Lie group homomorphism** is a smooth map

\[ \varphi : G \to H \quad \text{satisfying} \quad \varphi(gh) = \varphi(g)\varphi(h) \]

for all \( g \) and \( h \). For an element \( g \) in \( G \) we denote its derivative by \( d_g \varphi : T_gG \to T_{\varphi(g)}H \). In the special case of \( g = e \) (the identity in \( G \)) we obtain an isomorphism of Lie algebras:

\[ \varphi_* : g \to h, \quad \text{with} \quad \varphi_*([X,Y]) = [\varphi_*(X), \varphi_*(Y)], \]

and the important relation

\[ \varphi(e^X) = e^{\varphi_*(X)}. \]

### 1.3 Example: Parametrization of \( SO(3, \mathbb{R}) \)

As an example of a case where we can explicitly calculate the exponential of a matrix we take \( G = SO(3, \mathbb{R}) = \{ AA^T = \text{Id} \} \cap \{ \det A = 1 \} \). Its Lie algebra is given by \( g = so(3, \mathbb{R}) = \{ X + X^T = 0 \} \cap \{ \text{tr} X = 0 \} \) and is spanned by the elements

\[
F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

for which we have

\[ [F_1, F_2] = F_3, \quad [F_2, F_3] = F_1, \quad [F_3, F_1] = F_2. \]

By choosing this basis we obtain the linear map

\[ \mathbb{R}^3 \to so(3), \quad (x_1, x_2, x_3) \mapsto x_1 F_1 + x_2 F_2 + x_3 F_3, \]

which is in fact an isomorphism of Lie algebras if we take the standard cross-product on \( \mathbb{R}^3 \). For an element \( x \in \mathbb{R}^3 \) we denote its image under this map by \( A_x \). The eigenvalues of \( A_x \) are 0 (with eigenvector \( x \)) and \( \pm i|x| \). The geometric interpretation of the exponential of \( A_x \) is that of a rotation around the axis \( \mathbb{R}x \subset \mathbb{R}^3 \) by the angle \( |x| \). Using \( A_x^3 = -|x|^2 A_x \) and the standard series expansions of \( \sin \) and \( \cos \) we can simplify (1) and obtain

\[
e^{A_x} = \text{Id} + \frac{\sin |x|}{|x|} A_x + \frac{1 - \cos |x|}{|x|^2} A_x^2.
\]
1.4 Example: The “covering” $SU(2) \to SO(3)$

In the case $n = 2$ the group $SU(2)$ can be written as

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \bigg| |\alpha|^2 + |\beta|^2 = 1 \right\}.$$ 

Its Lie algebra $su(2)$ is given by

$$su(2) = \left\{ \begin{pmatrix} i\varphi & z \\ -\bar{z} & -i\varphi \end{pmatrix} \bigg| \varphi \in \mathbb{R}, z \in \mathbb{C} \right\}.$$ 

With the basis

$$E_1 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad E_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E_3 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

we have

$$su(2) = \mathbb{R}E_1 + \mathbb{R}E_2 + \mathbb{R}E_3$$

and

$$[E_1, E_2] = E_3, \quad [E_2, E_3] = E_1, \quad [E_3, E_1] = E_2.$$ 

(Observe that by setting $\sigma_k = 2/\imath E_k$ for $k = 1, 2, 3$, one obtains the Pauli matrices known in physics.) With the basis given above the Lie algebra $su(2)$ is isomorphic to $(\mathbb{R}^3, \times)$, and the group $GL(su(2))$ of invertible linear operators on $su(2)$ can in this way be identified with $GL(3, \mathbb{R})$.

The group $SU(2)$ acts on $su(2)$ by conjugation, i.e. for all $g \in SU(2)$ we have the map $X \mapsto gXg^{-1}$. Thus, we have the group homomorphism

$$SU(2) \to GL(su(2)) \cong GL(3, \mathbb{R}), \quad g \mapsto g(\cdot)g^{-1}.$$ 

One can prove that its image is exactly $SO(3, \mathbb{R}) \subset GL(3, \mathbb{R})$. Its kernel is $\{-\text{Id}, \text{Id}\}$. An explicit description of this homomorphism can be found in [DK].

1.5 The Derivative of the Exponential Mapping

We are now interested in the derivative of exp at a point $X \in g$. Before we can write down the general formula we need to introduce some notation: For $g \in G$ we denote the map $G \to G, x \mapsto gx$ by $L_g$. For $X \in g$ we have the linear map

$$\text{ad} \ X : g \to g, \quad Y \mapsto [X, Y].$$

Since this map is an operator on the finite-dimensional vector space $g$, the exponential $e^{\text{ad} \ X}$ can be defined in the usual way (see equation (1)).
In the case of general smooth manifolds $M$ and $N$ and a map $f : M \to N$, the derivative $d_p f$ at a point $p \in M$ goes from $T_p M$ to $T_{f(p)} N$. In our case, $M$ is simply the vector space $\mathfrak{g}$, so with $X \in \mathfrak{g}$ we can identify $T_X \mathfrak{g}$ with $\mathfrak{g}$ itself. The derivative of the exponential map is then given by

$$d_X \exp : \mathfrak{g} \to T_{\exp(X)} G$$

and

$$d_X \exp = d_e L_{\exp X} \circ \int_0^1 e^{-s \text{ad} X} ds. \quad (2)$$

An elementary proof of this formula is given is [Ha], chapter three. However, it can be shown that the derivative of the exponential map can be derived from more general principles, which is briefly discussed in appendix A.

If $G$ is a matrix Lie group then the above map $L_g : x \mapsto gx$ is linear. Hence, its derivative at the point $e \in G$ is the same as the map itself, i.e. $d_e L_g$ is the left-multiplication by $g$. So $d_e L_{\exp X}(Y) = e^X \cdot Y$, and (2) simplifies to

$$d_X \exp = e^X \int_0^1 e^{-s \text{ad} X} ds = e^X \sum_{k=0}^{\infty} \frac{(-\text{ad} X)^k}{(k+1)!} = e^X \frac{\text{Id} - e^{-\text{ad} X}}{\text{ad} X},$$

where the right-hand side of the last equality is actually defined by the left-hand side, because $\text{ad} X$ might not be invertible.

## 2 Coset Spaces and Homogenous Spaces

If $G$ is a general group (not necessarily a Lie group) and $K$ a subgroup of $G$, then we can form, for each element $g \in G$, the set $gK = \{ gk | k \in K \}$. The set of all such $gK$ is called the **coset space** and denoted by

$$G/K = \{ gK | g \in G \}.$$  

It is important to note that such a coset space is, in general, not a group anymore. Associated to a coset space $G/K$ we have the naturally defined map $\pi : G \to G/K$ assigning to each $g$ its coset $gK$. The group $G$ acts transitively on $G/K$ by left-multiplication, which we sometimes denote by the map $L_g : G/K \to G/K, \ xK \mapsto gxK$.

Note that since Lie algebras are vector spaces, they are in particular groups with regards to vector addition, so the above construction can also be applied to them. In such a setting, if $\mathfrak{k}$ is a subalgebra of a Lie algebra $\mathfrak{g}$, unless the subgroup $K$ is “normal” in $G$, which means that $gK = Kg$ for all $g \in G$. 

---

3Unless the subgroup $K$ is “normal” in $G$, which means that $gK = Kg$ for all $g \in G$. 

5
Figure 1: The existence of local coordinates in the example of the $S^1$-action on $\mathbb{C}^*$. The $S^1$-orbits are concentric circles around the origin. For a sufficiently small neighborhood we can always find a local slice, which parametrizes the orbits and, thus, defines local coordinates for the coset space $\mathbb{C}^*/S^1$.

then $\mathfrak{g}/\mathfrak{k}$ is the usual quotient vector space. In general, this quotient vector space is not a Lie algebra anymore.

Now if $G$ is a Lie group and $K$ is a closed subgroup it can be shown that the coset space $G/K$ can be endowed with the structure of a smooth manifold. We refer to such a space as a homogenous space.$^4$

2.1 Existence of Local Coordinates

The existence of smooth local coordinates on $G/K$ can be illustrated as follows: An open set in $G/K$ is by definition the image of an open set in $G$ under the map $\pi : G \to G/K$. If we take an open set $U$ in $G$, the image $\pi(U) \subset G/K$ can be interpreted as the set of all $K$-orbits intersecting $U$. It is a theorem that, for a sufficiently small neighborhood $U$, one can always find a local slice, i.e. a smooth submanifold of $U$ which parametrizes the orbits intersecting $U$ (see Figure 1 for an illustrative example where $G = GL(1, \mathbb{C}) = \mathbb{C}^* = \mathbb{C}\{0\}$ and $K = U(1) = S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$). Such a slice defines local coordinates for $\pi(U) \subset G/K$.

$^4$Homogenous spaces are important examples of abstractly defined smooth manifolds, i.e. manifolds that do not appear as subsets of some ambient space.
2.2 The Tangent Space

Since homogeneous spaces are manifolds we can talk about the tangent space at a point in $G/K$. At the point $eK \in G/K$ the tangent space can be expressed by the Lie algebras of $G$ and $K$:

$$T_{eK}(G/K) = \mathfrak{g}/\mathfrak{k}. \quad (3)$$

This can be seen as follows: The projection map $\pi : G \to G/K$ is surjective. So its derivative at the identity in $G$, which we denote by $\pi_* : T_eG \to T_{eK}(G/K)$, is also surjective. It is well-known from Linear Algebra that any surjective linear map $F : V \to W$ between vector spaces induces an isomorphism between $V/\ker F$ and $W$. Now the kernel of $\pi_*$ is precisely $\mathfrak{k}$. Together with $T_eG = \mathfrak{g}$ this yields equation (3).

3 Symmetric Spaces

A symmetric space is a homogeneous space $G/K$ where the subgroup $K$ has two additional properties:

1. $K$ is a compact\(^5\) subgroup of $G$, and

2. there exists an involution $\theta : G \to G$ (i.e. a Lie group homomorphism satisfying $\theta^2 = \text{Id}$) with $\text{Fix}(\theta) \subset K \subset \text{Fix}(\theta)$,

where $\text{Fix}(\theta) = \{\theta(g) = g\}$ is the fixed point set of $\theta$ and $\text{Fix}^\circ(\theta)$ is the topological component of $\text{Fix}(\theta)$ containing the identity element of $G$. In many examples we simply have $K = \text{Fix}(\theta)$.

Examples

- $G = SL(n, \mathbb{C}), K = SU(n), \theta(A) = (A^\dagger)^{-1}$.
- $G = SU(n), K = SO(n), \theta(A) = \bar{A}$.

From the definition of a symmetric space it follows that every symmetric space can be equipped with a Riemannian metric. This metric is not unique. However, the geodesics defined by such a Riemannian metric are in fact unique. See [He] or [CE] for details.

---

\(^5\)A matrix Lie group $G$ is **compact** if convergent sequences in $G$ have their limit in $G$, and if there exists a constant $C$ such that for all $A \in G$, $|A_{ij}| \leq C$ for all $1 \leq i, j \leq n$. For example, the groups $O(n)$, $SO(n)$, and $SU(n)$ are compact.
3.1 The Cartan Decomposition

From the second property in the definition of a symmetric space it follows that the Lie algebra \( \mathfrak{k} \) of \( K \) is given as the \((+1)\)-eigenspace of \( \theta^* : \mathfrak{g} \to \mathfrak{g} \). The \((-1)\)-eigenspace of \( \theta_\star \) is usually denoted by \( \mathfrak{p} \) and, together with \( \mathfrak{k} \), yields the Cartan decomposition of \( \mathfrak{g} \)

\[ \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}. \]

Examples

- For \( G = \text{SL}(n, \mathbb{C}) \), \( K = \text{SU}(n) \), we have \( \theta_\star(X) = -X^\dagger \) and, thus, \( \mathfrak{k} = \{ X = -X^\dagger \} \) and \( \mathfrak{p} = \{ X = X^\dagger \} \).

- For \( G = \text{SU}(n) \), \( K = \text{SO}(n) \), it follow \( \theta_\star(X) = \overline{X} \) and, thus,

\[ \mathfrak{k} = \{ X \in \mathfrak{su}(n) \mid X = \overline{X} \} = \{ X \in \mathbb{R}^{n \times n} \mid X = -X^T \} \]

and \( \mathfrak{p} = \{ X \in \mathbb{R}^{n \times n} \mid X = X^T \} \).

Note that \( \mathfrak{p} \) is a canonically defined complement of \( \mathfrak{k} \). One can think of it as being “perpedicular” to \( \mathfrak{k} \). Since \( \theta_\star \) preserves the bracket operation, one can easily prove the following important inclusions:

\[ [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k} \quad \text{and} \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}. \quad (4) \]

From the Cartan decomposition it follows that \( \mathfrak{p} \) is canonically isomorphic to the quotient \( \mathfrak{g}/\mathfrak{k} \), which in turn is the tangent space of \( G/K \) at the point \( eK \). Thus

\[ T_{eK}(G/K) = \mathfrak{g}/\mathfrak{k} \cong \mathfrak{p} \quad (5) \]

3.2 The Exponential Map for Symmetric Spaces

Since symmetric spaces have a close relation to Lie groups, one can also define an exponential map for them:

\[ \exp : \mathfrak{p} \to G/K, \quad \Exp = \pi \circ \exp \mid \mathfrak{p} \quad (6) \]

The derivative of \( \Exp \) at a point \( X \in \mathfrak{p} \) is the map \( d_X\Exp : \mathfrak{p} \to T_{\Exp X}G/K \). In view of (6) it can be calculated by using the derivative of (Lie-theoretic) \( \exp \) and applying the chain rule. Furthermore, we can simplify the resulting expression by using equation (4) and \( d_e\pi(\mathfrak{k}) = 0 \) and obtain

\[ d_X\Exp = d_{eK}\mathcal{L}_{\exp} X \circ \sum_{k=0}^{\infty} \frac{(-\ad X)^{2k}}{(2k + 1)!}, \]

8
where $\tilde{L}_g(xK) = gxK$ is defined as in section 2. Using

$$\frac{\sinh(x)}{x} = \sum_{k=0}^{\infty} \frac{(-x)^{2k}}{(2k + 1)!}$$

we get an expression of $d_X\text{Exp}$ which is perhaps easier to remember:

$$d_X\text{Exp} = d_{eK} \tilde{L}_{\exp X} \circ \frac{\sinh \text{ad} X}{\text{ad} X}.$$ 

### 3.3 The Exponential Map and Geodesics

If $M$ is a general Riemannian manifold one can always define a local diffeomorphism from the tangent space at a point $p \in M$ into $M$ which maps straight lines passing through the origin in $T_pM$ to geodesics in $M$. This map is usually called the Riemannian exponential map and denoted by $\text{Exp}$. See [He] for details.

In section 2 it was remarked that geodesics on a symmetric space are independent of the choice of the respective Riemannian metric. Thus, the Riemannian exponential map is unique in such a context. It can be shown that it can be directly expressed by the usual Lie-theoretic exponential map.

So, by using (5), we have a relation between straight lines in $\mathfrak{p}$ passing through the origin and geodesics in $G/K$ (see figure 2).
A  A Variational Formula for time-dependent Vector Fields

Let $M$ be a smooth manifold and $v_\varepsilon$ a smoothly parametrized family of vector fields on $M$. For each $v_\varepsilon$ we have an associated flow $\Phi^t_\varepsilon$ which we assume to be globally defined. The derivative of $\Phi^t_\varepsilon$ with respect to $\varepsilon$ can be expressed by the following variational formula:

$$\frac{d}{d\varepsilon} \Phi^t_\varepsilon(x) = \int_0^t d_{\Phi^s_\varepsilon(x)} \Phi^{t-s}_\varepsilon v_\varepsilon \frac{d}{d\varepsilon} (\Phi^s_\varepsilon(x)) \, ds \in T_{\Phi^t_\varepsilon M}. \quad (7)$$

In the context of a general Lie group $G$ with Lie algebra $\mathfrak{g}$ we can apply the above formula to the vector fields defined by $X + \varepsilon Y$, where $X$ and $Y$ are elements of the Lie algebra of $G$ and $\varepsilon$ is a real number. The associated flow is then given by the exponential map

$$\Phi^s_\varepsilon(x) = \exp(s(X + \varepsilon Y))x, \quad x \in G.$$  

Applying this to (7) and then simplifying yields the general formula for the derivative of the exponential map:

$$d_X \exp = d_\varepsilon L_{\exp X} \circ \int_0^1 e^{-s \text{ad}X} \, ds.$$

More details can be found in [DK] and [He].

B  Acknowledgements

We would like to thank the members of the “Seminar der Doktoranden des SFB/TR-12, Trace Formulas and Symmetric Spaces” for their help and suggestions.
C  Notation

\( \bar{A} \)  Complex conjugate of \( A \)

\( A^T \)  The transpose of \( A \)

\( A^\dagger \)  The adjoint of \( A \), i.e. \( \bar{A}^T \)

\( \text{Id} \)  The identity matrix

\( GL(n, \mathbb{C}) \) \{ \( A \in \mathbb{C}^{n \times n} \mid \det A \neq 1 \} \)

\( SL(n, \mathbb{C}) \) \{ \( A \in \mathbb{C}^{n \times n} \mid \det A = 1 \} \)

\( U(n) \) \{ \( A \in GL(n, \mathbb{C}) \mid AA^\dagger = \text{Id} \} \)

\( SU(n) \) \( U(n) \cap SL(n, \mathbb{C}) \)

\( O(n) \) \{ \( A \in GL(n, \mathbb{R}) \mid AA^T = \text{Id} \} \)

\( SO(n) \) \( O(n) \cap SL(n, \mathbb{R}) \)

\( G \)  A general (matrix) Lie group

\( G^\circ \)  The topological component of \( G \) which contains the identity element

\( e \)  The identity element of a Lie group

\( e^X \)  The exponential of a matrix \( X \), i.e. \( e^X = \sum_{k=0}^{\infty} X^k/k! \)

\( T_gG \)  The tangent space of \( G \) at a point \( g \in G \)

\( \mathfrak{g} \)  The Lie algebra of \( G \), i.e. \( \mathfrak{g} = T_eG \)

\([X,Y] \)  The commutator of two Lie algebra elements,

i.e. \( [X,Y] = XY - YX \)

\( L_g \)  The map defined by left multiplication by \( g \in G \),

i.e. \( L_g(x) = gx \)

\( d_pf \)  The differential of a smooth map \( f : M \rightarrow N \)
between manifolds.

\( \varphi_* \)  The differential of a Lie group homomorphism at
the point \( e \in G \), i.e. \( \varphi_* = d_e\varphi \)

\( \text{ad} X \)  The map defined by \( (\text{ad} X)(Y) = [X,Y] \)

\( \text{Fix}(\theta) \)  The fixed point set of the involution \( \theta \)

\( \text{Fix}(\theta)^\circ \)  The topological component of \( \text{Fix}(\theta) \) containing the identity.

References


