# Symmetric Spaces Toolkit

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# 1 Lie Groups and Lie Algebras

## 1.1 Matrix Lie Groups

A subset $G$ of $GL(n,\mathbb{C})$ that is topologically closed and also closed under the group operations (i.e. if $A, B \in G$ then $AB^{-1} \in G$ as well) is called a **matrix Lie group**. It can be shown (see e.g. [Co]) that such a matrix Lie group is automatically a differentiable sub-manifold of $GL(n,\mathbb{C})$. Note that this definition also includes zero-dimensional matrix groups like $\{\mathrm{Id}, -\mathrm{Id}\} \subset GL(n)$.

In this article all matrix Lie groups are supposed to be **reductive**, which – in our context – means that for each element $A$ of the group its adjoint $A^\dagger = \bar{A}^T$ is also an element of the group.[1]

In the context of matrix Lie groups we have the usual exponential map

$$\exp : \mathbb{C}^{n\times n} \to GL(n,\mathbb{C}), \quad X \mapsto e^X = \sum_{k=0}^{\infty} \frac{X^k}{k!}\,. \tag{1}$$

For a given matrix Lie group $G$ we consider the set

$$\mathfrak{g} = \left\{ X \in GL(n) \,\middle|\, e^{tX} \in G \,\text{for all}\, t \in \mathbb{R} \right\},$$

and call it the **Lie algebra** of $G$. In this definition the map $t \mapsto e^{tX}$ is a curve in $G$ which passes through the identity $\mathrm{Id} \in G$. Since

$$\frac{d}{dt}\Big|_{t=0} e^{tX} = X$$

we can think of $\mathfrak{g}$ as being the set of tangent vectors at the identity $\mathrm{Id} \in G$, i.e.

$$\mathfrak{g} = T_{\mathrm{Id}} G.$$

Together with the commutator $[X, Y] = XY - YX$ one can show (see for example [Ha]) that $\mathfrak{g}$ is also an Lie algebra in the abstract sense.[2]

### Examples of Lie groups and their Lie algebras

- $G = SL(n,\mathbb{C}) = \{A \in \mathbb{C}^{n\times n} \mid \det A = 1\}$, $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C}) = \{X \in \mathbb{C}^{n\times n} \mid \operatorname{tr} X = 0\}$.

[1] This is not the abstract definition of reductivity, which is more complicated (see for example [Hu]). However, it can be shown that a matrix Lie group which fulfills the definition of reductivity given here is also reductive in the abstract sense.

[2] Recall that an (abstract) **Lie algebra** is a vector space with an additional product structure that is bilinear, skew-symmetric, and fulfills the Jacobi identity, i.e. $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$.

- $G = SU(n) = \{AA^\dagger = \mathrm{Id}\} \cap \{\det A = 1\}$, $\mathfrak{g} = \mathfrak{su}(n) = \{X + X^\dagger = 0\} \cap \{\operatorname{tr} X = 0\}$.

## 1.2 Lie Group Homomorphisms

A **Lie group homomorphism** is a smooth map

$$\varphi : G \to H \quad \text{satisfying} \quad \varphi(gh) = \varphi(g)\varphi(h)$$

for all $g$ and $h$. For an element $g$ in $G$ we denote its derivative by $d_g\varphi : T_gG \to T_{\varphi(g)}H$. In the special case of $g = e$ (the identity in $G$) we obtain an isomorphism of Lie algebras:

$$\varphi_* : \mathfrak{g} \to \mathfrak{h}, \quad \text{with} \quad \varphi_*([X,Y]) = [\varphi_*(X), \varphi_*(Y)],$$

and the important relation

$$\varphi(e^X) = e^{\varphi_*(X)}.$$

## 1.3 Example: Parametrization of $SO(3,\mathbb{R})$

As an example of a case where we can explicitly calculate the exponential of a matrix we take $G = SO(3,\mathbb{R}) = \{AA^T = \mathrm{Id}\} \cap \{\det A = 1\}$. Its Lie algebra is given by $\mathfrak{g} = \mathfrak{so}(3,\mathbb{R}) = \{X + X^T = 0\} \cap \{\operatorname{tr} X = 0\}$ and is spanned by the elements

$$F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad F_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad F_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for which we have

$$[F_1, F_2] = F_3, \quad [F_2, F_3] = F_1, \quad [F_3, F_1] = F_2.$$

By choosing this basis we obtain the linear map

$$\mathbb{R}^3 \to \mathfrak{so}(3), \quad (x_1, x_2, x_3) \mapsto x_1F_1 + x_2F_2 + x_3F_3,$$

which is in fact an isomorphism of Lie algebras if we take the standard cross-product on $\mathbb{R}^3$. For an element $x \in \mathbb{R}^3$ we denote its image under this map by $A_x$. The eigenvalues of $A_x$ are 0 (with eigenvector $x$) and $\pm i|x|$. The geometric interpretation of the exponential of $A_x$ is that of a rotation around the axis $\mathbb{R}x \subset \mathbb{R}^3$ by the angle $|x|$. Using $A_x^3 = -|x|^2 A_x$ and the standard series expansions of sin and cos we can simplify (1) and obtain

$$e^{A_x} = \mathrm{Id} + \frac{\sin|x|}{|x|}A_x + \frac{1 - \cos|x|}{|x|^2}A_x^2.$$

## 1.4 Example: The "covering" $SU(2) \to SO(3)$

In the case $n = 2$ the group $SU(2)$ can be written as

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \;\middle|\; |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

Its Lie algebra $\mathfrak{su}(2)$ is given by

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} i\varphi & z \\ -\bar{z} & -i\varphi \end{pmatrix} \;\middle|\; \varphi \in \mathbb{R},\, z \in \mathbb{C} \right\}.$$

With the basis

$$E_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad E_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

we have

$$\mathfrak{su}(2) = \mathbb{R}E_1 + \mathbb{R}E_2 + \mathbb{R}E_3$$

and

$$[E_1, E_2] = E_3, \quad [E_2, E_3] = E_1, \quad [E_3, E_1] = E_2.$$

(Observe that by setting $\sigma_k = 2/iE_k$ for $k = 1, 2, 3$, one obtains the **Pauli matrices** known in physics.) With the basis given above the Lie algebra $\mathfrak{su}(2)$ is isomorphic to $(\mathbb{R}^3, \times)$, and the group $GL(\mathfrak{su}(2))$ of invertible linear operators on $\mathfrak{su}(2)$ can in this way be identified with $GL(3, \mathbb{R})$.

The group $SU(2)$ acts on $\mathfrak{su}(2)$ by conjugation, i.e. for all $g \in SU(2)$ we have the map $X \mapsto gXg^{-1}$. Thus, we have the group homomorphism

$$SU(2) \to GL(\mathfrak{su}(2)) \cong GL(3, \mathbb{R}), \quad g \mapsto g(\cdot)g^{-1}.$$

One can prove that its image is exactly $SO(3, \mathbb{R}) \subset GL(3, \mathbb{R})$. Its kernel is $\{-\mathrm{Id}, \mathrm{Id}\}$. An explicit description of this homomorphism can be found in [DK].

## 1.5 The Derivative of the Exponential Mapping

We are now interested in the derivative of exp at a point $X \in \mathfrak{g}$. Before we can write down the general formula we need to introduce some notation: For $g \in G$ we denote the map $G \to G$, $x \mapsto gx$ by $L_g$. For $X \in \mathfrak{g}$ we have the linear map

$$\operatorname{ad} X : \mathfrak{g} \to \mathfrak{g}, \quad Y \mapsto [X, Y].$$

Since this map is an operator on the finite-dimensional vector space $\mathfrak{g}$, the exponential $e^{\operatorname{ad} X}$ can be defined in the usual way (see equation (1)).

In the case of general smooth manifolds $M$ and $N$ and a map $f : M \to N$, the derivative $d_p f$ at a point $p \in M$ goes from $T_pM$ to $T_{f(p)}N$. In our case, $M$ is simply the vector space $\mathfrak{g}$, so with $X \in \mathfrak{g}$ we can identify $T_X\mathfrak{g}$ with $\mathfrak{g}$ itself. The derivative of the exponential map is then given by

$$d_X \exp : \mathfrak{g} \to T_{\exp(X)}G$$

and

$$d_X \exp = d_e L_{\exp X} \circ \int_0^1 e^{-s \operatorname{ad} X} \, ds. \tag{2}$$

An elementary proof of this formula is given is [Ha], chapter three. However, it can be shown that the derivative of the exponential map can be derived from more general principles, which is briefly discussed in appendix A.

If $G$ is a matrix Lie group then the above map $L_g : x \mapsto gx$ is linear. Hence, its derivative at the point $e \in G$ is the same as the map itself, i. e. $d_eL_g$ is the left-multiplication by $g$. So $d_e L_{\exp X}(Y) = e^X \cdot Y$, and (2) simplifies to

$$d_X \exp = e^X \int_0^1 e^{-s \operatorname{ad} X} \, ds = e^X \sum_{k=0}^{\infty} \frac{(-\operatorname{ad} X)^k}{(k+1)!} = e^X \frac{\operatorname{Id} - e^{-\operatorname{ad} X}}{\operatorname{ad} X},$$

where the right-hand side of the last equality is actually defined by the left-hand side, because $\operatorname{ad} X$ might not be invertible.

# 2 Coset Spaces and Homogenous Spaces

If $G$ is a general group (not neccessarily a Lie group) and $K$ a subgroup of $G$, then we can form, for each element $g \in G$, the set $gK = \{gk \mid k \in K\}$. The set of all such $gK$ is called the **coset space** and denoted by

$$G/K = \{gK \mid g \in G\}.$$

It is important to note that such a coset space is, in general, not a group anymore.[3] Associated to a coset space $G/K$ we have the naturally defined map $\pi : G \to G/K$ assigning to each $g$ its coset $gK$. The group $G$ acts transitively on $G/K$ by left-multiplication, which we sometimes denote by the map $\tilde{L}_g : G/K \to G/K$, $xK \mapsto gxK$.

Note that since Lie algebras are vector spaces, they are in particular groups with regards to vector addition, so the above construction can also be applied to them. In such a setting, if $\mathfrak{k}$ is a subalgebra of a Lie algebra $\mathfrak{g}$,

[3]Unless the subgroup$K$ is "normal" in $G$, which means that $gK = Kg$ for all $g \in G$.

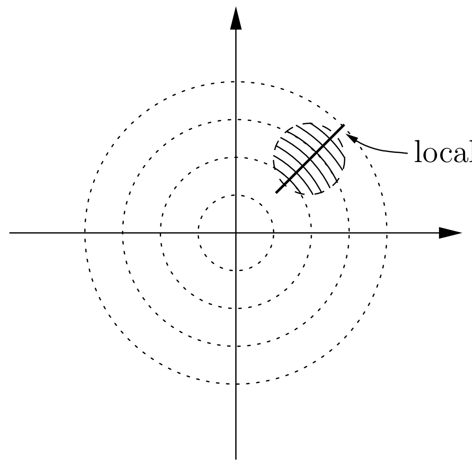


Figure 1: The existence of local coordinates in the example of the $S^1$-action on $\mathbb{C}^*$. The $S^1$-orbits are concentric circles around the origin. For a sufficiently small neighborhood we can always find a local slice, which parametrizes the orbits and, thus, defines local coordinates for the coset space $\mathbb{C}^*/S^1$.

then $\mathfrak{g}/\mathfrak{k}$ is the usual quotient vector space. In general, this quotient vector space is not a Lie algebra anymore.

Now if $G$ is a Lie group and $K$ is a closed subgroup it can be shown that the coset space $G/K$ can be endowed with the structure of a smooth manifold. We refer to such a space as a **homogenous space**.[4]

## 2.1 Existence of Local Coordinates

The existence of smooth local coordinates on $G/K$ can be illustrated as follows: An open set in $G/K$ is by definition the image of an open set in $G$ under the map $\pi : G \to G/K$. If we take an open set $U$ in $G$, the image $\pi(U) \subset G/K$ can be interpreted as the set of all $K$-orbits intersecting $U$. It is a theorem that, for a sufficiently small neighborhood $U$, one can always find a **local slice**, i.e. a smooth submanifold of $U$ which parametrizes the orbits intersecting $U$ (see Figure 1 for an illustrative example where $G = GL(1, \mathbb{C}) = \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ and $K = U(1) = S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$). Such a slice defines local coordinates for $\pi(U) \subset G/K$.

[4]Homogenous spaces are important examples of abstractly defined smooth manifolds, i.e. manifolds that do not appear as subsets of some ambient space.

## 2.2 The Tangent Space

Since homogenous spaces are manifolds we can talk about the tangent space at a point in $G/K$. At the point $eK \in G/K$ the tangent space can be expressed by the Lie algebras of $G$ and $K$:

$$T_{eK}\big(G/K\big) = \mathfrak{g}/\mathfrak{k}. \tag{3}$$

This can be seen as follows: The projection map $\pi : G \to G/K$ is surjective. So its derivative at the identity in $G$, which we denote by $\pi_* : T_eG \to T_{eK}\big(G/K\big)$, is also surjective. It is well-known from Linear Algebra that any surjective linear map $F : V \to W$ between vector spaces induces an isomorphism between $V/\ker F$ and $W$. Now the kernel of $\pi_*$ is precisely $\mathfrak{k}$. Together with $T_eG = \mathfrak{g}$ this yields equation (3).

# 3 Symmetric Spaces

A **symmetric space** is a homogeneous space $G/K$ where the subgroup $K$ has two additional properties:

1. $K$ is a *compact*[5] subgroup of $G$, and
2. there exists an **involution** $\theta : G \to G$ (i.e. a Lie group homomorphism satisfying $\theta^2 = \mathrm{Id}$) with $\mathrm{Fix}(\theta)^\circ \subset K \subset \mathrm{Fix}(\theta)$,

where $\mathrm{Fix}(\theta) = \{\theta(g) = g\}$ is the fixed point set of $\theta$ and $\mathrm{Fix}^\circ(\theta)$ is the topological component of $\mathrm{Fix}(\theta)$ containing the identity element of $G$. In many examples we simply have $K = \mathrm{Fix}(\theta)$.

**Examples**

- $G = SL(n, \mathbb{C})$, $K = SU(n)$, $\theta(A) = (A^\dagger)^{-1}$.
- $G = SU(n)$, $K = SO(n)$, $\theta(A) = \bar{A}$.

From the definition of a symmetric space it follows that every symmetric space can be equipped with a Riemannian metric. This metric is not unique. However, the geodesics defined by such a Riemannian metric are in fact unique. See [He] or [CE] for details.

[5] A matrix Lie group $G$ is **compact** if convergent sequences in $G$ have their limit in $G$, and if there exists a constant $C$ such that for all $A \in G$, $|A_{ij}| \leq C$ for all $1 \leq i, j \leq n$. For example, the groups $O(n)$, $SO(n)$, and $SU(n)$ are compact.

## 3.1 The Cartan Decomposition

From the second property in the definition of a symmetric space it follows that the Lie algebra $\mathfrak{k}$ of $K$ is given as the $(+1)$-eigenspace of $\theta_* : \mathfrak{g} \to \mathfrak{g}$. The $(-1)$-eigenspace of $\theta_*$ is usually denoted by $\mathfrak{p}$ and, together with $\mathfrak{k}$, yields the **Cartan decomposition** of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

**Examples**

- For $G = SL(n, \mathbb{C})$, $K = SU(n)$, we have $\theta_*(X) = -X^\dagger$ and, thus, $\mathfrak{k} = \{X = -X^\dagger\}$ and $\mathfrak{p} = \{X = X^\dagger\}$.
- For $G = SU(n)$, $K = SO(n)$, it follow $\theta_*(X) = \bar{X}$ and, thus,

  $$\mathfrak{k} = \{X \in \mathfrak{su}(n) \,|\, X = \bar{X}\} = \{X \in \mathbb{R}^{n\times n} \,|\, X = -X^T\}$$

  and $\mathfrak{p} = \{X \in \mathbb{R}^{n\times n} \,|\, X = X^T\}$.

Note that $\mathfrak{p}$ is a canonically defined complement of $\mathfrak{k}$. One can think of it as being "perpedicular" to $\mathfrak{k}$. Since $\theta_*$ preserves the bracket operation, one can easily prove the following important inclusions:

$$[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k} \quad \text{and} \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}. \tag{4}$$

From the Cartan decomposition it follows that $\mathfrak{p}$ is canonically isomorphic to the quotient $\mathfrak{g}/\mathfrak{k}$, which in turn is the tangent space of $G/K$ at the point $eK$. Thus

$$T_{eK}\big(G/K\big) = \mathfrak{g}/\mathfrak{k} \cong \mathfrak{p} \tag{5}$$

## 3.2 The Exponential Map for Symmetric Spaces

Since symmetric spaces have a close relation to Lie groups, one can also define an exponential map for them:

$$\mathrm{Exp} : \mathfrak{p} \to G/K, \quad \mathrm{Exp} = \pi \circ \exp|\mathfrak{p} \tag{6}$$

The derivative of Exp at a point $X \in \mathfrak{p}$ is the map $d_X\mathrm{Exp} : \mathfrak{p} \to T_{\mathrm{Exp}X}G/K$. In view of (6) it can be calculated by using the derivative of (Lie-theoretic) exp and applying the chain rule. Furthermore, we can simplify the resulting expression by using equation (4) and $d_e\pi(\mathfrak{k}) = 0$ and obtain

$$d_X\mathrm{Exp} = d_{eK}\tilde{L}_{\exp X} \circ \sum_{k=0}^{\infty} \frac{(-\operatorname{ad} X)^{2k}}{(2k+1)!},$$

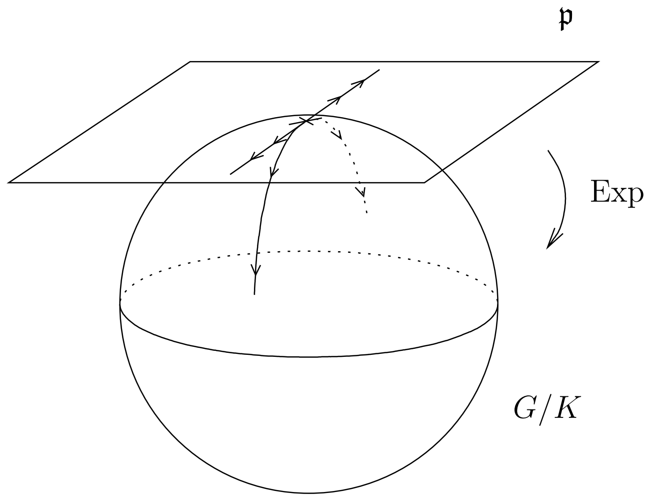


Figure 2: Straight lines in $\mathfrak{p}$ passing through the origin are mapped to geodesics in $G/K$ by Exp.

where $\tilde{L}_g(xK) = gxK$ is defined as in section 2. Using

$$\frac{\sinh(x)}{x} = \sum_{k=0}^{\infty} \frac{(-x)^{2k}}{(2k+1)!}$$

we get an expression of $d_X\mathrm{Exp}$ which is perhaps easier to remember:

$$d_X\mathrm{Exp} = d_{eK}\tilde{L}_{\exp X} \circ \frac{\sinh \operatorname{ad} X}{\operatorname{ad} X}.$$

## 3.3 The Exponential Map and Geodesics

If $M$ is a general Riemannian manifold one can always define a local diffeomorphism from the tangent space at a point $p \in M$ into $M$ which maps straight lines passing through the origin in $T_pM$ to geodesics in $M$. This map is usually called the *Riemannian exponential map* and denoted by Exp. See [He] for details.

In section 2 it was remarked that geodesics on a symmetric space are independet of the choice of the respective Riemannian metric. Thus, the Riemannian exponential map is unique in such a context. It can be shown that it can be directly expressed by the usual Lie-theoretic exponential map.

So, by using (5), we have a relation between straight lines in $\mathfrak{p}$ passing through the origin and geodesics in $G/K$ (see figure 2).

# A A Variational Formula for time-dependent Vector Fields

Let $M$ be a smooth manifold and $v_\varepsilon$ a smoothly parametrized family of vector fields on $M$. For each $v_\varepsilon$ we have an associated flow $\Phi_\varepsilon^t$ which we assume to be globally defined. The derivative of $\Phi_\varepsilon^t$ with respect to $\varepsilon$ can be expressed by the following variational formula:

$$\frac{d}{d\varepsilon}\Phi_\varepsilon^t(x) = \int_0^t d_{\Phi_\varepsilon^s(x)}\Phi_\varepsilon^{t-s}\frac{v_\varepsilon}{d\varepsilon}(\Phi_\varepsilon^s(x))\,ds \in T_{\Phi_\varepsilon^t}M. \tag{7}$$

In the context of a general Lie group $G$ with Lie algebra $\mathfrak{g}$ we can apply the above formula to the vector fields defined by $X + \varepsilon Y$, where $X$ and $Y$ are elements of the Lie algebra of $G$ and $\varepsilon$ is a real number. The associated flow is then given by the exponential map

$$\Phi_\varepsilon^s(x) = \exp(s(X + \varepsilon Y))x, \quad x \in G.$$

Applying this to (7) and then simplifying yields the general formula for the derivative of the exponential map:

$$d_X \exp = d_e L_{\exp X} \circ \int_0^1 e^{-s \operatorname{ad} X}\,ds.$$

More details can be found in [DK] and [He].

# B Acknowledgements

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# C Notation

| | |
|---|---|
| $\bar{A}$ | Complex conjugate of $A$ |
| $A^T$ | The transpose of $A$ |
| $A^\dagger$ | The adjoint of $A$, i.e. $\bar{A}^T$ |
| $\mathrm{Id}$ | The identity matrix |
| $GL(n, \mathbb{C})$ | $\{A \in \mathbb{C}^{n\times n} \mid \det A \neq 1\}$ |
| $SL(n, \mathbb{C})$ | $\{A \in \mathbb{C}^{n\times n} \mid \det A = 1\}$ |
| $U(n)$ | $\{A \in GL(n, \mathbb{C}) \mid AA^\dagger = \mathrm{Id}\}$ |
| $SU(n)$ | $U(n) \cap SL(n, \mathbb{C})$ |
| $O(n)$ | $\{A \in GL(n, \mathbb{R}) \mid AA^T = \mathrm{Id}\}$ |
| $SO(n)$ | $O(n) \cap SL(n, \mathbb{R})$ |
| $G$ | A general (matrix) Lie group |
| $G^\circ$ | The topological component of $G$ which contains the identity element |
| $e$ | The identity element of a Lie group |
| $e^X$ | The exponential of a matrix $X$, i.e. $e^X = \sum_{k=0}^{\infty} X^k/k!$ |
| $T_gG$ | The tangent space of $G$ at a point $g \in G$ |
| $\mathfrak{g}$ | The Lie algebra of $G$, i.e. $\mathfrak{g} = T_eG$ |
| $[X, Y]$ | The commutator of two Lie algebra elements, i.e. $[X, Y] = XY - YX$ |
| $L_g$ | The map defined by left multiplication by $g \in G$, i.e. $L_g(x) = gx$ |
| $d_pf$ | The differential of a smooth map $f : M \to N$ beween manifolds. |
| $\varphi_*$ | The differential of a Lie group homomorphism at the point $e \in G$, i.e. $\varphi_* = d_e\varphi$ |
| $\mathrm{ad}\, X$ | The map definied by $(\mathrm{ad}\, X)(Y) = [X, Y]$ |
| $\mathrm{Fix}(\theta)$ | The fixed point set of the involution $\theta$ |
| $\mathrm{Fix}(\theta)^\circ$ | The topological component of $\mathrm{Fix}(\theta)$ containing the identity. |

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